Faculty of Science, Technology, Engineering and Mathematics
M208 Pure mathematics

M208

TMA 07 2019J

(Revision) **Cut-off date 7 May 2020**

You can submit your TMA either by post to your tutor or electronically as a PDF file by using the University's online TMA/EMA service.

Before starting work on the TMA, please read the document *Student guidance for preparing and submitting TMAs*, available from the 'Assessment' tab of the M208 website.

In the wording of the questions:

- *write down*, *list* or *state* means 'write down without justification' (unless otherwise stated)
- *find*, *determine*, *calculate*, *derive*, *evaluate* or *solve* means 'show all your working'
- *prove*, *show*, *deduce* or *verify* means 'justify each step'
- *sketch* means 'sketch without justification' and *describe* means 'describe without justification' (both unless otherwise stated).

In particular, if you use a definition, result or theorem to go from one line to the next, then you must state clearly which fact you are using – for example, you could quote the relevant unit and page, or give a Handbook reference. Remember that when you use a theorem, you must demonstrate that all the conditions of the theorem are satisfied.

The number of marks assigned to each part of a question is given in the right-hand margin, to give you a rough indication of the amount of time that you should spend on each part.

Your work should be written in a good mathematical style, as demonstrated by the exercise and worked exercise solutions in the study texts. You should explain your solutions carefully, using appropriate notation and terminology, defining any symbols that you introduce, and writing in proper sentences.

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3.1

The questions in this TMA are similar in style to some of the questions that you will be asked in the exam. There are questions on each book of the module.

Introduction questions (Book A)

Question 1 – 8 marks

Given that all the roots of the polynomial

$$p(x) = x^3 - 3x + 2$$

are integers, determine $p(x)$ as a product of linear factors. *[8]*

Question 2 – 8 marks

Use mathematical induction to prove the following statement about the derivatives of the function $f(x) = \sin(5x)$:

$$f^{(2n-1)}(x) = (-1)^{n-1}5^{2n-1}\cos(5x), \quad \text{for } n = 1, 2, \ldots.$$ *[8]*

Linear algebra questions (Book C)

Question 3 – 9 marks

This question concerns the system of linear equations

$$\begin{aligned} x + 3y - 2z &= 0, \\ 2x + y + z &= 20, \\ y + 6z &= 3. \end{aligned}$$

(a) Write down the augmented matrix of this system of linear equations. *[1]*

(b) Find the row-reduced form of the matrix that you wrote down in part (a). *[6]*

(c) Solve the system of equations by using the row-reduced form of the augmented matrix. *[2]*

Question 4 – 9 marks

Let t be the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (2x - 4y + 6z, x - 2y + 3z). \end{aligned}$$

(a) Find a basis for $\operatorname{Im} t$, and give a geometric description of $\operatorname{Im} t$. *[6]*

(b) Find $\operatorname{Ker} t$, describing it geometrically. *[3]*

Group theory questions (Books B and E)

Question 5 – 11 marks

Consider the subset $G = \{1, 2, 4, 8, 11, 16\}$ of the group $(U_{21}, \times_{21})$.

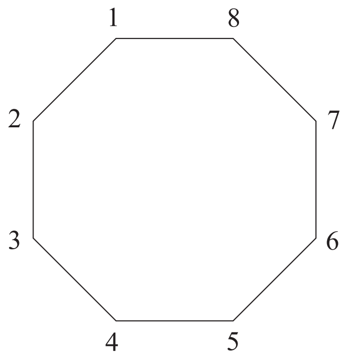
(a) Show that G is a cyclic subgroup of $(U_{21}, \times_{21})$. *[3]*

(b) Determine all the subgroups of G, justifying your answer. *[5]*

(c) Write down an isomorphism ϕ that maps $(G, \times_{21})$ to $(\mathbb{Z}_6, +_6)$. *[3]*

Question 6 – 11 marks

This question concerns the symmetry group G of the regular octagon shown below.



(a) Let g be the anticlockwise rotation through $\pi/2$ about the centre of the octagon, and let h be the reflection in the line through the vertices of the octagon at locations 2 and 6.

(i) Write g, g^2 and h in cycle form, using the labelling of the vertex locations shown above.

(ii) Express the conjugate $g \circ h \circ g^{-1}$ in cycle form, and describe it geometrically. *[5]*

(b) Consider the elements $j = (1\ 5)(2\ 6)(3\ 7)(4\ 8)$ and $k = (1\ 8)(2\ 7)(3\ 6)(4\ 5)$ of G.

(i) Are j and k conjugate in S_8? Justify your answer, and if they are conjugate, find an element of S_8 that conjugates j to k.

(ii) Are j and k conjugate in G? Justify your answer, and if they are conjugate, find an element of G that conjugates j to k.

(iii) Describe geometrically the conjugacy class of G that contains k. *[6]*

Question 7 – 11 marks

The set

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}$$

forms a group under matrix multiplication, with identity element

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This question concerns the mapping ϕ defined by

$$\begin{aligned} \phi : G &\longrightarrow G \\ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} &\longmapsto \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

(a) Prove that ϕ is a homomorphism. *[3]*

(b) Find $\operatorname{Ker}\phi$ and $\operatorname{Im}\phi$. *[4]*

(c) Show that the quotient group $G/\operatorname{Ker}\phi$ is isomorphic to $(\mathbb{R}, +)$. *[4]*

Analysis questions (Books D and F)

Question 8 – 11 marks

Determine whether each of the following sequences (a_n) is convergent, stating the limit of the sequence if it exists. You should name any result or rule that you use.

(a) $a_n = \dfrac{2^n + 3(4^n) + 5}{3^n - 5(4^n)}$ *[3]*

(b) $a_n = \dfrac{\cos(n\pi) + n^2}{3n^2 + 2^n}$ *[4]*

(c) $a_n = \dfrac{n^2\cos(n\pi) + 2n}{3n^2 - 1}$ *[4]*

Question 9 – 11 marks

(a) Prove that the function

$$h(x) = -2x - 1$$

is uniformly continuous on $\mathbb{R}$. *[4]*

(b) Let f be the function defined by

$$f(x) = \begin{cases} -e^x, & x < 0, \\ -2x - 1, & x \geq 0. \end{cases}$$

(i) Sketch the graph of f. *[1]*

(ii) Prove that f is continuous but not differentiable at 0. *[6]*

Question 10 – 11 marks

(a) Show that

$$\lim_{x \to 1} \frac{\cos\left(\frac{\pi x}{2}\right)}{x-1}$$

exists, and determine its value. [*6*]

(b) Prove that

$$\left| \int_{\pi}^{2\pi} \frac{3\cos x - 3}{x^2}\, dx \right| \leq \frac{6}{\pi}.$$ [*5*]